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A Cluster Expansion for Stochastic Lattice Fields

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A Langevin equation of Landau–Ginzburg type for the stochastic dynamics of a scalar field on a lattice is studied. A cluster expansion is developed for this problem which converges for large mass. As a consequence, one establishes uniformly in the volume: (a) exponential decay of correlations in space and time, and (b) exponential approach to equilibrium for a class of nearby initial distributions.

KEY WORDS: Landau–Ginzburg equation; cluster expansion.

1. INTRODUCTION

Let Λ be a finite d-dimensional lattice which for definiteness take to be a toroidal lattice $(\mathbb{Z}/L\mathbb{Z})^d$ for some integer L. For fields $\varphi \in \mathbb{R}^\Lambda$ consider an action of the form

$$S(\varphi) = \sum_{x \in \Lambda} \left[\frac{1}{2} \varphi_x((-\Delta + m^2)\varphi)_x + \frac{\lambda}{4} \varphi_x^4 \right] \tag{1.1}$$

where Δ is the Laplacian on Λ and λ and m^2 are positive parameters. Then there is an associated Langevin equation for the stochastic dynamics of φ given by

$$\begin{aligned} \frac{d\varphi}{dt} &= -\frac{1}{2} \nabla S + \eta \\ &= -\frac{1}{2}(-\Delta + m^2)\varphi - \frac{1}{2} \lambda \varphi^3 + \eta \end{aligned} \tag{1.2}$$

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Here $\eta = \eta_x(t)$ is a white noise random variable with covariance

$$E(\eta_x(t)\, \eta_{x'}(t')) = \delta_{xx'}\, \delta(t-t')$$

One expects (and we will see) that $\exp[-S(\varphi)]\, d\varphi$ is a stationary distribution for solutions of this equation and that all other initial distributions tend to it. It is called the equilibrium or Gibbs state.

This model can be taken to describe the time evolution of an order parameter for a statistical mechanical system, for example, the magnetization. Then it would be called time-dependent Landau–Ginzburg theory. This application is described in Hohenberg and Halperin.[1] (In their terminology it is model A.)

On the other hand, the equilibrium state (or a continuum version of it) can be though of as φ^4 Euclidean quantum field theory. Then t is just a parameter and is not interpreted as time. One can adopt the strategy of studying the equilibrium state by investigating the full process. This is known as stochastic quantization. The method (which extends to many other field theories) has some substantial advantages, particularly for numerical work. This application is described in Parisi[2] and Damgaard and Hüffel.[4]

In this paper we study solutions of (1.2) for a class of initial distributions close to equilibrium. For this class (described precisely in Section 2.3) and for m sufficiently large we show that all the correlation functions have exponential decay in x and t uniformly in the volume Λ. For example, for the two-point function

$$\begin{aligned} &|E(\varphi_x(t)\, \varphi_{x'}(t')) - E(\varphi_x(t))\, E(\varphi_{x'}(t'))| \\ &\qquad \leqslant \mathcal{O}(1) \exp[-a(|x-x'| + |t-t'|)] \end{aligned} \tag{1.3}$$

As an application, we show an exponential approach to equilibrium. Of course, bounds like (1.3) are central to many infinite-volume questions.

There is a substantial mathematical literature on related problems, especially on stochastic Ising models in which the state space (i.e., the X in the configuration space X^Λ) is finite. A sampling of work with continuous state space related to that given here can be found in refs. 5–8.

There are also treatments of continuum models ($\Lambda \subset \mathbb{R}^d$) in low dimension for exactly the present model, i.e., stochastic φ^4_d field theories. These are due to Faris and Jona-Lasinio (for $d=1$)[9] and Jona-Lasinio and Mitter (for $d=2$).[10]

2. SOLUTIONS

2.1. The Linear Case

Let us start by finding solutions of (1.2) for $\lambda = 0$. The equation can be written in the standard form

$$d\varphi = -\tfrac{1}{2}D\varphi\, dt + dB$$
$$D = (-\Delta + m^2) \tag{2.1}$$

where B is a Brownian motion in $\mathbb{R}^{\Lambda}$, that is, a family $\{B_x(t)\}$, $x \in \Lambda$, $t \in [0, \infty)$, of Gaussian random variables such that

$$E(B_x(t)\, B_{x'}(s)) = \delta_{x,x'} \min(t, s)$$

This is the equation for an Orenstein–Uhlenbeck process with drift term $-\tfrac{1}{2}D\varphi$.

A solution φ with initial point $\chi \in \mathbb{R}^{\Lambda}$ is given by the stochastic integral

$$\hat{\varphi}(t) = \exp^{-Dt/2}\chi + \int_0^t e^{-D(t-s)/2}\, dB(s) \tag{2.2}$$

This process has continuous trajectories. It can be realized on the space of continuous functions

$$\Omega = C^0([0, \infty), \mathbb{R}^{\Lambda})$$

as the coordinate function $(\varphi(t))(\omega) = \omega(t)$, $\omega \in \Omega$, if we supply Ω with the measure

$$\mu_\chi^0(A) = E(1_A(\hat{\varphi})) \tag{2.3}$$

where $A \in \mathscr{F}$, the σ-algebra generated by the $\varphi(t)$.

Next we average over the initial point defining μ^0 on Ω by

$$\mu^0(A) = \int \mu_\chi^0(A)\, d\nu^0(\chi) \tag{2.4}$$

where $\nu^0 \equiv \nu_{D^{-1}}$ is the Gaussian measure on $\mathbb{R}^{\Lambda}$ with covariance D^{-1}. A short calculation shows that for μ^0 the coordinate function is Gaussian with covariance

$$C_{x,x'}(t, t') = [D^{-1} \exp(-\tfrac{1}{2}D|t - t'|)]_{xx'} \tag{2.5}$$

Accordingly, we also write $\mu^0 = \mu_C$. The process is seen to be stationary, confirming that μ^0 is an equilibrium distribution. If we also allow negative times, then $C_{x,x'}(t, t')$ can be regarded as the kernel of operator $C = (-\partial^2/\partial t^2 + \frac{1}{4}D^2)^{-1}$.

The decay properties of C will be important for later developments. We have that for any $a \geqslant 1$ there is an m_0 so that for $m \geqslant m_0$

$$|C_{x,x'}(t, t')| \leqslant \mathcal{O}(m^{-2}) \exp[-a(|x-x'| + |t-t'|)] \tag{2.6}$$

This bound can first be established for $(x, t) \in \mathbb{Z}^d \times \mathbb{R}$ by a contour deformation in Fourier transform space. The toroidal case $(x, t) \in \Lambda \times \mathbb{R}$ is expressed as a periodization of the infinite-volume case, and the bound carries over.

2.2. The General Case

For $\lambda > 0$, write Eq. (1.2) as

$$\begin{aligned} d\varphi &= (-\tfrac{1}{2}D\varphi + b(\varphi))\, dt + dB \\ (b(\varphi))_x &= -\tfrac{1}{2}\lambda\varphi_x^3 \end{aligned} \tag{2.7}$$

We want to find a measure μ_χ on Ω such that the coordinate function on Ω is a solution of the equation with initial point χ. The Girsanov–Cameron–Martin formula gives such a measure as a perturbation of the measure μ_χ^0 for $\lambda = 0$. If E_χ^0 is the expectation for μ_χ^0, then

$$\mu_\chi(A) = E_\chi^0(1_A Z(t)) \tag{2.8}$$

where

$$Z(t) = \exp\left[\int_0^t (b(\varphi(s)), d\bar{\varphi}(s)) - \frac{1}{2}\int_0^t \|b(\varphi(s))\|^2\, ds\right] \tag{2.9}$$

Here the inner product and norm are in $\mathbb{R}^\Lambda$ and $\bar{\varphi}(t) \equiv \varphi(t) + \int_0^t \frac{1}{2}D\varphi(s)\, ds$ is a Brownian motion. For $A \in \mathscr{F}_s$ [the σ-algebra generated by $\varphi(s')$, $0 \leqslant s' \leqslant s$], one can take any t satisfying $t \geqslant s$.

If b were bounded, this would be quite standard (see any book on stochastic differential equations, for example, Friedman[11] or Stroock and Varadhan[12]). Since it is not, some further justification is needed.

Let us start by finding an alternate expression for $Z(t)$. Let

$$\begin{aligned} P(\varphi) &= \sum_{x \in \Lambda} \frac{\lambda}{4} \varphi_x^4 \\ W(\varphi) &= \sum_{x \in \Lambda} \left[-\frac{3}{4}\lambda\varphi_x^2 + \frac{\lambda}{4}\varphi_x^3(D\varphi)_x + \frac{\lambda^2}{8}\varphi_x^6\right] \end{aligned} \tag{2.10}$$

Then the claim is that

$$Z(t)=\exp\left[\frac{1}{2}P(\varphi(0))-\frac{1}{2}P(\varphi(t))-\int_0^t W(\varphi(s))\,ds\right] \tag{2.11}$$

To see this, note that $-\frac{1}{2}\int_0^t \|b\|^2$ gives the sixth-order term in W. For the other term, use Ito's formula:

$$\begin{aligned}\int_0^t (b(\varphi(s)), d\bar{\varphi}(s)) &= -\frac{1}{2}\int_0^t ((\nabla P)(\varphi(s)), d\bar{\varphi}(s)) \\ &= \frac{1}{2}P(\varphi(0))-\frac{1}{2}P(\varphi(t))+\frac{1}{4}\int_0^t ((\Delta-(D\varphi)\cdot\nabla)P)(\varphi(s))\,ds\end{aligned}$$

The last term gives the quadratic and quartic terms in W.

Lemma 2.1:

(a) $W(\varphi)\geqslant\lambda\left(\frac{m^2}{8}-d\right)\left(\sum_{x\in\Lambda}\varphi_x^4\right)-\frac{9}{4}\left(\frac{\lambda}{m^2}\right)|\Lambda|$

(b) $$|Z(t)|\leqslant\exp\left[\sum_{x\in\Lambda}\left\{\frac{\lambda}{8}[\varphi_x^4(0)-\varphi_x^4(t)]-\lambda\left(\frac{m^2}{8}-d\right)\int_0^t\varphi_x^4(s)\,ds\right\}+\frac{9}{4}\left(\frac{\lambda}{m^2}\right)t\right]$$

Proof. (a) In $D=(-\Delta+m^2)$ the lattice Laplacian has matrix elements

$$(-\Delta)_{xy}=\left\{\begin{array}{ll}2d & x=y\\ -1 & |x-y|=1\\ 0 & \text{otherwise}\end{array}\right\}$$

Correspondingly, there is a nonlocal contribution to W with absolute value

$$\left|\frac{\lambda}{4}\sum_{|x-y|=1}\varphi_x^3\varphi_y\right|\leqslant\frac{\lambda}{4}\sum_{|x-y|=1}(\varphi_x^4+\varphi_y^4)\leqslant\lambda d\sum_x\varphi_x^d \tag{2.12}$$

The rest of W is local and has the form

$$\sum_x \left[-\frac{3}{4}\lambda\varphi_x^2 + \frac{\lambda}{4}(m^2+2d)\varphi_x^4 + \frac{\lambda^2}{8}\varphi_x^6 \right]$$

$$\geqslant \sum_x \left(\frac{\lambda m^2}{8}\varphi_x^4 - \frac{9}{4}\frac{\lambda}{m^2} \right) \tag{2.13}$$

where we use

$$-\frac{3}{4}\lambda\varphi^2 + \lambda\frac{m^2}{8}\varphi^4 \geqslant -\frac{9}{4}\left(\frac{\lambda}{m^2}\right)$$

Combining these gives the result. Part (b) follows from (a).

Theorem 2.2:

(a) $\mu_\chi(A) = E_\chi^0(1_A Z(t))$ defines a probability measure on $(\Omega, \mathscr{F}_s)$ which is independent of t for $t \geqslant s$.

(b) For this measure the coordinate function $\varphi(\cdot)$ on Ω solves (2.7) in the sense that it solves the associated martingale problem.

Proof. (a) By Ito's formula for the exponential function and the process $\int b \cdot d\bar{\varphi} - \frac{1}{2}\int \|b\|^2\, dt$, we find

$$Z(t) = 1 + \int_0^t Z(s)(b(\varphi(s)), d\bar{\varphi}(s)) \tag{2.14}$$

Now $b(\varphi(s))$ is square integrable and by the lemma $Z(s)$ is bounded $[\varphi(0) = \chi]$. Then $Z(s)\, b(\varphi(s))$ is square-integrable and nonanticipating, so (2.14) shows that $Z(t)$ is a martingale: $E_\chi^0(Z(t) \mid \mathscr{F}_s) = Z(s)$ for $t \geqslant s$. Taking the expectation for $s=0$ gives $E_\chi^0(Z(t)) = 1$, so μ_χ is a probability measure. For $A \in \mathscr{F}_s$ and $t > s$, $E_\chi^0(1_A Z(t) \mid \mathscr{F}_s) = 1_A Z(s)$ and taking the expectation shows that the measure is independent of t.

(b) We must show that for $\theta \in \mathbb{R}^\Lambda$

$$X^\theta(t) \equiv \exp\left\{ \left(\theta, \varphi(t) - \varphi(0) - \int_0^t \left[-\tfrac{1}{2} D\varphi(s) + b(\varphi(s)) \right] ds \right) - \frac{1}{2}\|\theta\|^2 t \right\} \tag{2.15}$$

is a martingale for μ_χ.[12] This will follow if we show that $Z^\theta(t) = X^\theta(t)\, Z(t)$ is a martingale for μ_χ^0. But we have

$$Z^\theta(t) = \exp\left[\int_0^t (b(\varphi(s)) + \theta, d\bar{\varphi}(s)) - \frac{1}{2}\int_0^t \|b(\varphi(s)) + \theta\|^2\, ds \right] \tag{2.16}$$

This is a martingale by the argument of part (a), once we show it is bounded. However,

$$|X^\theta(t)| \leqslant \exp\left[\|\theta\|_\infty \sum_x \left\{ |\varphi_x(0)| + |\varphi_x(t)| + \int_0^t \left[\mathcal{O}(1)\,|\varphi_x(s)| + \frac{\lambda}{2}\,|\varphi_x(s)|^3 \right] ds \right\} \right]$$

and combining this with the bound on $Z(t)$ from the lemma shows that $Z^\theta(t)$ is bounded.

2.3. Initial Distributions

Let us consider initial distributions given by measures v on $\mathbb{R}^\Lambda$ of the form

$$v(B) = v^0(B\exp(-P-Q))/v^0(\exp(-P-Q)) \tag{2.17}$$

Here P is given by (2.10) as before and gives the equilibrium measure; see below. Q gives the deviation from equilibrium and is taken to have the form

$$Q(\varphi) = \lambda \sum_{x\in\Lambda} q(\varphi_x) \tag{2.18}$$

where q is a polynomial which is either bounded below or at least such that $\frac{1}{8}\varphi^4 + q(\varphi)$ is bounded below. We could also allow some nonhomogeneity in x in Q, but for simplicity stick with (2.18).

Averaging μ_χ over the initial point χ with weight v, we define measures μ on Ω by

$$\mu(A) = \int \mu_\chi(A)\, dv(\chi) \tag{2.19}$$

By (2.4), (2.8), (2.11), and (2.17) we find for $A \in \mathscr{F}_s$

$$\mu(A) = \left(\int 1_A e^{-V_t}\, d\mu_C\right)\Big/\left(\int e^{-V_t}\, d\mu_C\right) \tag{2.20}$$

where for $t \geqslant s$

$$V_t(\varphi) = \frac{1}{2}P(\varphi(0)) + Q(\varphi(0)) + \frac{1}{2}P(\varphi(t)) + \int_0^t W(\varphi(s))\, ds \tag{2.21}$$

If F is $\mathscr{F}_s$-measurable, then the expectation with respect to μ would be, for $t \geqslant s$,

$$E(F) = \left(\int F e^{-V_t} \, d\mu_C \right) \Big/ \left(\int e^{-V_t} \, d\mu_C \right) \tag{2.22}$$

These are the objects of interest, particularly the correlation functions for which $F(\varphi) = \varphi_{x_1}(t_1) \cdots \varphi_{x_n}(t_n)$.

Note that the measure μ_C, and these integrals in particular, which are defined on the space $\Omega = C^0([0, \infty), \mathbb{R}^A)$, could also be regarded as defined on the space $\Omega = C^0(\mathbb{R}, \mathbb{R}^A)$, as is done in the rest of this section, or as defined on $\Omega = C^0([0, t], \mathbb{R}^A)$, as is done in Section 3.

If $Q = 0$, we use the notation V_t^*, μ^*, E^* for the equilibrium quantities. The next result shows that μ^*, E^* are indeed stationary.

First we need some definitions. Let $\mathscr{F}_{a,b}$ be the σ-algebra generated by $\varphi(u)$, $a \leqslant u \leqslant b$. Let T_t be the time translation operator on Ω defined by $(T_t \omega)(s) = \omega(s - t)$ and let θ be the reflection operator $(\theta \omega)(t) = \omega(-t)$. Finally, define

$$V_{s,t}^* = \frac{1}{2} P(\varphi(s)) + \frac{1}{2} P(\varphi(t)) + \int_s^t W(\varphi(u)) \, du$$

so that $V_{0,t}^* = V_t^*$.

Proposition 2.3. Let F be $\mathscr{F}_{a,b}$ measurable $(0 \leqslant a \leqslant b)$ and μ_C integrable. Then:

(a) $\int F \exp(-V_{s,t}^*) \, d\mu_C$ is independent of s, t for $0 \leqslant s \leqslant a \leqslant b \leqslant t$

(b) $\int (F \circ T_{-u}) \exp(-V_{0,t}^*) \, d\mu_C$ is independent of u for $0 \leqslant u \leqslant t - b$

Proof. (a) We want to replace $V_{s,t}^*$ by $V_{a,b}^*$. If $s = 0$, we already know that we can replace t by b. We reduce the general case to this. For $0 < s \leqslant a$, we have

$$\begin{aligned} \int F \exp(-V_{s,t}^*) \, d\mu_C &= \int (F \circ T_s) \exp(-V_{0,t-s}^*) \, d\mu_C \\ &= \int (F \circ T_s) \exp(-V_{0,b-s}^*) \, d\mu_C \\ &= \int F \exp(-V_{s,b}^*) \, d\mu_C \end{aligned}$$

Here we use that μ_C is translation invariant and that $F \circ T_s$ is $\mathscr{F}_{a-s,b-s}$ measurable and hence $\mathscr{F}_{0,b-s}$ measurable.

To shorten the left endpoint, we use

$$\int F \exp(-V^*_{s,b})\, d\mu_C = \int (F \circ T_b \circ \theta) \exp(-V^*_{0,b-s})\, d\mu_C$$

$$= \int (F \circ T_b \circ \theta) \exp(-V^*_{0,b-a})\, d\mu_C$$

$$= \int F \exp(-V^*_{a,b})\, d\mu_C$$

Here we use that μ_C is reflection invariant and that $F \circ T_b \circ \theta$ is $\mathscr{F}_{0,b-a}$ measurable.

(b) By part (a), since $F \circ T_{-u}$ is $\mathscr{F}_{a+u,b+u}$ measurable and since 0, $u \leqslant a+u$ and $b+u \leqslant t$, $t+u$:

$$\int (F \circ T_{-u}) \exp(-V^*_{0,t})\, d\mu_C = \int (F \circ T_{-u}) \exp(-V^*_{u,t+u})\, d\mu_C$$

$$= \int F \exp(-V^*_{0,t})\, d\mu_C$$

Remark. Consider the equilibrium correlation functions at equal times $E^*(\varphi_{x_1}(s) \cdots \varphi_{x_n}(s))$. By translation invariance we can take $s=0$ and then replace $V^*_{0,t}$ by $V^*_{0,0} = P(\varphi(0))$. For the time zero fields μ_C is just ν_{D-1} and so

$$E^*(\varphi_{x_1}(s) \cdots \varphi_{x_n}(s))$$
$$= \int \varphi_{x_1} \cdots \varphi_{x_n} \exp[-P(\varphi)]\, d\nu_{D-1}(\varphi) \Big/ \int \exp[-P(\varphi)]\, d\nu_{D-1}(\varphi)$$

The right side is a φ^4 field theory, and this is a fundamental identity of stochastic quantization.

3. EXPANSIONS

3.1. Mayer Expansion

We want to study long-distance properties of integrals of the form $\int F \exp(-V_T)\, d\mu_C$. The method is a general technique known as cluster expansions, which expresses such integrals as sums of local pieces. General references are refs. 13, 14, and 18. Our expansion is not quite standard because we have both discrete and continuous variables in the underlying space, and because there is nonlocality both in the potential V_T (due to D) and in the measure $d\mu_C$ (due to C).

We begin by breaking the nonlocality in V_T with a Mayer expansion of $\exp(-V_T)$. Beginning with the expression (2.21), we write

$$V_T = \sum_I V_I = \sum_I \left(\sum_x V_{I,x} + \sum_b V_{I,b} \right) \tag{3.1}$$

where the sum is over unit intervals I in $[0, T]$ (now with T integral), points $x \in \Lambda$, and nearest neighbor pairs (bonds) b in Λ.

$V_{I,x}$ is the local piece and is given by

$$\begin{aligned} V_{I,x} = &\int_I \left[-\frac{3}{4} \lambda \varphi_x^2(s) + \frac{\lambda}{4}(m^2 + 2d)\, \varphi_x^4(s) + \frac{\lambda^2}{8} \varphi_x^6(s) \right] ds \\ &+ \int_I \left[\frac{\lambda}{8} \varphi_x^4(s) + \lambda q(\varphi_x(s)) \right] \delta(s)\, ds + \int_I \left[\frac{\lambda}{8} \varphi_x^4(s) \right] \delta(s-T)\, ds \end{aligned} \tag{3.2}$$

For each bond $b = (x, y)$, $|x - y| = 1$ we have the nonlocal piece

$$V_{I,b} = \int_I \frac{\lambda}{4} \varphi_x^3(s)\, \varphi_y(s)\, ds \tag{3.3}$$

The Mayer expansion for V_I is

$$\exp(-V_I) = \prod_{x \in \Lambda} \exp(-V_{I,x}) \sum_{\{b_\alpha\}} \prod_\alpha [\exp(-V_{I,b_\alpha}) - 1]$$

where the sum is over collections $\{b_\alpha\}$ of bonds in Λ. If we group together the terms in this sum into connected clusters, we have

$$\exp(-V_I) = \sum_{\{Y_\beta\}} \prod_\beta \rho(I \times Y_\beta)$$

where the sum is over partitions $\{Y_\beta\}$ of Λ and for $Y \subset \Lambda$

$$\rho(I \times Y) = \prod_{x \in Y} \exp(-V_{I,x}) \sum_{\{b_\alpha\} \to Y} \prod_\alpha [\exp(-V_{I,b_\alpha}) - 1] \tag{3.4}$$

where the sum is over $\{b_\alpha\}$ connecting Y. As a special case, we have $\rho(I \times \{x\}) = \exp(-V_{I,x})$.

Taking the product over I, we have

$$\exp(-V_T) = \sum_{\{U_\gamma\}} \prod_\gamma \rho(U_\gamma) \tag{3.5}$$

where the sum is over partitions $\{U_\gamma\}$ of $[0, T] \times \Lambda$ into sets of the form $U = I \times Y$ with Y connected in Λ.

The basic estimate on ρ is

Lemma 3.1. For m^2 sufficiently large, $\lambda \leqslant 1$, and $|Y| \geqslant 2$

$$|\rho(I \times Y)| \leqslant [\mathcal{O}(m^{-2})]^{|Y|-1} \tag{3.6}$$

Proof. We use

$$|\exp(-V_{I,b}) - 1| \leqslant \frac{8d}{m^2} \exp\left(\frac{m^2}{8d} |V_{I,b}|\right)$$

and (2.12) to obtain

$$\left|\prod_{\alpha} [\exp(-V_{I,b_\alpha}) - 1]\right| \leqslant \left(\frac{8d}{m^2}\right)^{|Y|-1} \exp\left[\frac{\lambda m^2}{8} \sum_{x \in Y} \int_I \varphi_x^4(s)\, ds\right] \tag{3.7}$$

On the other hand, as in (2.13),

$$\left|\prod_{x \in Y} \exp(-V_{I,x})\right| \leqslant \exp\left[-\frac{\lambda m^2}{8} \sum_{x \in Y} \int_I \varphi_x^4(s)\, ds\right] \exp[\mathcal{O}(1)\, |Y|] \tag{3.8}$$

Thus,

$$\rho(I \times Y) \leqslant \sum_{\{b_\alpha\} \to Y} [\mathcal{O}(m^{-2})]^{|Y|-1} \leqslant [\mathcal{O}(m^{-2})]^{|Y|-1}$$

the last step since there are at most $[\mathcal{O}(1)]^{|Y|} \leqslant [\mathcal{O}(1)]^{|Y|-1}$ terms in the sum.

3.2. Modified Gaussian Measures

The next step will be to expand the measure μ_C into measures with covariances which do not couple certain regions of space and time. We start by explaining the elementary step in this expansion.

Define a paved set in $[0, T] \times \Lambda$ to be a set expressible as an union of unit lines $I \times \{x\}$. For paved sets S, S' define

$$C(S, S')_{x,x'}(t, t') = 1_S(x, t)\, C_{x,x'}(t, t')\, 1_{S'}(x', t') \tag{3.9}$$

and $C(S) = C(S, S)$. We want to consider a Gaussian process with covariances $C(S)+C(\sim S)$ which decouple S and $\sim S$. More generally, consider convex combinations of such covariances, such as

$$C_s = sC + (1-s)[C(S)+C(\sim S)], \qquad s \in [0, 1] \tag{3.10}$$

which interpolates between $C_0 = C(S)+C(\sim S)$ and $C_1 = C$.

These covariances are not smooth unless we alter the space. Define

$$[0, T]' = [0, 1] \cup (1, 2) \cup \cdots \cup (T-1, T]$$

If S (and $\sim S$) are paved sets in $[0, T]' \times \Lambda$, then $(C_s)_{x,x'}(t, t')$ is a continuous positive-definite function on $([0, T]' \times \Lambda) \times ([0, T]' \times \Lambda)$ and so defines a Gaussian process with this covariance. Furthermore, C_s is smooth enough to admit continuous sample paths, so the process may be realized as the coordinate function on

$$\Omega' = C^0([0, T]' \times \Lambda) \cong C^0([0, T]', \mathbb{R}^\Lambda)$$

Let μ_{C_s} denote the associated measure on this space. Note that a special case of all this is the original process for C (with integral times deleted except 0 and T).

The difference between μ_{C_1} and μ_{C_0} is expressed by

$$\begin{aligned} &\int F\, d\mu_{C_1} - \int F\, d\mu_{C_0} \\ &\quad = \int_0^1 ds\, d/ds \left(\int F\, d\mu_{C_s} \right) \\ &\quad = \int_0^1 ds \int \frac{1}{2} (dC_s/ds) \circ \Delta_\varphi F\, d\mu_{C_s} \\ &\quad = \int_0^1 ds \int \frac{1}{2} (C_1 - C_0) \circ \Delta_\varphi F\, d\mu_{C_s} \end{aligned} \tag{3.11}$$

Here for any C a symbol like $C \circ \Delta_\varphi$ stands for the formal differential operator

$$C \circ \Delta_\varphi = \int dt\, dt' \sum_{x, x'} C_{x,x'}(t, t')\, \partial/\partial\varphi_x(t)\, \partial/\partial\varphi_{x'}(t') \tag{3.12}$$

We use (3.11) when F is a sum of terms of the form

$$\left[\int p_1(\varphi(t_1),..., \varphi(t_n))\, f(t_1,..., t_n)\, dt_1 \cdots dt_n \right] p_2(\varphi(0))\, p_3(\varphi(T)) e^{-V_T} \tag{3.13}$$

where p_1, p_2, p_3 are polynomials, and f is bounded. When C is also bounded, we claim that $C \circ \Delta_\varphi$ makes sense on this class of functions and in fact preserves the class [so that iteration of (3.11) is possible]. Applying Δ_φ formally introduces δ-functions. For example, to pick some typical pieces of V_T, if $G(\varphi) = \sum_x \varphi_x(0)^4$, then

$$\partial G/\partial\varphi_x(t) = 4\varphi_x(0)^3\,\delta(t)$$

$$\partial^2 G/\partial\varphi_x(t)\,\partial\varphi_{x'}(t') = 12\varphi_x(0)^2\,\delta_{x,x'}\,\delta(t)\,\delta(t')$$

or if $G(\varphi) = \sum_x \int_0^T \varphi_x(s)^4\,ds$,

$$\partial G/\partial\varphi_x(t) = 4\varphi_x(t)^3$$

$$\partial^2 G/\partial\varphi_x(t)\,\partial\varphi_{x'}(t') = 12\varphi_x(t)^2\,\delta_{x,x'}\,\delta(t-t')$$

The δ-functions are immediately evaluated in the integral defining $C \circ \Delta_\varphi$, and in this way one sees that $(C \circ \Delta_\varphi)F$ is again in the class (3.13).

To prove (3.11), one can first prove it for functions F depending on a finite number of variables by an explicit computation. The general case can be established by approximating the integrals in (3.3) by Riemann sums and then taking limits. For details in a case with two continuous variables see Dimock and Glimm.[15]

In applying (3.11), it is useful to note that the term $\int F\,d\mu_{C_0}$ may factor. Let $\Omega(S) = C^0(S)$. Since S and $\sim S$ are disconnected in $[0, T]' \times \Lambda$, we have $\Omega' \cong \Omega(S) \times \Omega(\sim S)$. Under this identification, $d\mu_{C(S)+C(\sim S)} \cong d\mu_{C(S)} \times d\mu_{C(\sim S)}$. Thus, if $F = F(S)\,F(\sim S)$ with $F(S)$ localized in S, etc., then

$$\int_{\Omega'} F\,d\mu_{C(S)+C(\sim S)} = \int_{\Omega(S)} F(S)\,d\mu_{C(S)} \cdot \int_{\Omega(\sim S)} F(\sim S)\,d\mu_{C(\sim S)} \tag{3.14}$$

3.3. Expansion of μ_C

Suppose now we are given a partition $\{U_\gamma\}$ of $[0, T]' \times \Lambda$ into connected paved sets as in Section 3.1, and suppose F is a function which factors across the partition, i.e., $F = \prod_\gamma F(U_\gamma)$ with $F(U_\gamma)$ localized in U_γ. We want to expand $\int F\,d\mu_C$ in localized pieces.

Let U_1 be the partition element which contains the first line $I \times \{x\}$ relative to some fixed ordering of unit lines. Apply (3.11) with $S = U_1$ and

$$C_{s_1} = s_1 C + (1 - s_1)[C(U_1) + C(\sim U_1)]$$

We have

$$\frac{1}{2}(\partial C_{s_1}/\partial s_1) \circ \Delta_\varphi = C(\sim U_1, U_1) \circ \Delta_\varphi = \sum_{U_2 \neq U_1} C(U_2, U_1) \circ \Delta_\varphi$$

and using also (3.14) we have

$$\int F\,d\mu_C=\left[\int F(U_1)\,d\mu_{C(U_1)}\right]\left[\int F(\sim U_1)\,d\mu_{C(\sim U_1)}\right]$$
$$+\sum_{U_2\neq U_1}\int_0^1 ds_1\,C(U_2,U_1)\circ\Delta_\varphi F\,d\mu_{C_{s_1}}$$

Now for each U_2 the function $C(U_2, U_1)\circ\Delta_\varphi F$ factors across $S_2=U_1\cup U_2$, and hence so will the integral if we break C_{s_1} on S_2 with

$$C_{s_1,s_2}=s_2C_{s_1}+(1-s_2)[C_{s_1}(S_2)+C_{s_1}(\sim S_2)]$$

In the error term we have

$$\frac{1}{2}(\partial C_{s_1,s_2}/\partial s_2)\circ\Delta_\varphi=C_{s_1}(\sim S_2,S_2)\circ\Delta_\varphi$$
$$=\sum_{U_3\neq U_1,U_2}[C_{s_1}(U_3,U_1)+C_{s_1}(U_3,U_2)]\circ\Delta_\varphi$$

Continuing, we get a sequence $U_1, U_2,..., U_n$ whose union is S_n and covariances

$$C_{s_1,...,s_n}=s_nC_{s_1,...,s_{n-1}}+(1-s_n)[C_{s_1,...,s_{n-1}}(S_n)+C_{s_1,...,s_{n-1}}(\sim S_n)] \tag{3.15}$$

For each $\{U_\gamma\}$-paved set X containing U_1 we group together the terms in the expansion for which $S_n=X$. Then the expansion has the form (see, for example, ref. 16)

$$\int F\,d\mu_C=\sum_X \bar{K}_F(X)\left[\int F(\sim X)\,d\mu_{C(\sim X)}\right] \tag{3.16}$$

Here $F(X)=\prod_\gamma F(U_\gamma)$, the product over the $\{U_\gamma\}$ in X. The quantity $\bar{K}_F(X)$ is defined by

$$\bar{K}_F(X)=\sum_{(U_1,...,U_n)}\sum_\eta\int ds\,f(\eta,s)\int\prod_{j=2}^n[C(U_j,U_{\eta(j)})\circ\Delta_\varphi]\,F(X)\,d\mu_{C_s(X)} \tag{3.17}$$

Here the first sum is over orderings $(U_1,..., U_n)$ of the $\{U_\gamma\}$ in X, but with U_1 always the first in the base ordering. If X has only the single element U_1, then $\bar{K}_F(X)=\int F(X)\,d\mu_{C(X)}$. The second sum is a sum over functions

η from $(2,..., n)$ to $(1,..., n-1)$ with $\eta(j) < j$; these can be thought of as the tree graphs on n vertices. Finally, $s = (s_1,..., s_{n-1})$, $C_s = C_{s_1,...,s_{n-1}}$, and

$$f(\eta, s) = \prod_{j=2}^{n} s_{j-2} s_{j-1} \cdots s_{\eta(j)}$$

with a factor 1 if $\eta(j) = j - 1$.

Now we iterate (3.16), beginning by decoupling the first partition element in $\sim X$. This yields

$$\int F \, d\mu_C = \sum_{\{X_i\}} \prod_i \bar{K}_F(X_i) \tag{3.18}$$

where the sum is over partitions $\{X_i\}$ into $\{U_\gamma\}$-paved sets X_i.

3.4. The Full Expansion

Now let F be a polynomial in $\varphi_x(t)$ that factors over paved sets, for example, a monomial. Combining the Mayer expansion (3.5) and the μ_C expansion (3.18), we have

$$\begin{aligned} \int F \exp(-V_T) \, d\mu_C &= \sum_{\{U_\gamma\}} \int F \prod_\gamma \rho(U_\gamma) \, d\mu_C \\ &= \sum_{\{U_\gamma\}} \sum_{\{X_i\}} \prod_i \bar{K}_{\rho F}(X_i) \end{aligned}$$

where, for each $\{U_\gamma\}$, $\rho = \prod_\gamma \rho(U_\gamma)$. Changing the order of the sums, we have a sum over $\{U_\gamma\}$ finer than or equal to $\{X_i\}$ which factors over the $\{X_i\}$. This leads to

$$\int F \exp(-V_T) \, d\mu_C = \sum_{\{X_i\}} \prod_i K_F(X_i) \tag{3.19}$$

where the sum is over all partitions $\{X_i\}$ of $[0, T]' \times \Lambda$ into paved sets. The $K_F(X)$ may be written as

$$\begin{aligned} K_F(X) = \sum_{\{U_\gamma\}} \sum_{(U_1,..., U_n)} \sum_\eta \int ds \, f(\eta, s) \int \prod_{j=2}^{n} [C(U_j, U_{\eta(j)}) \circ \Delta_\varphi] \\ \times \prod_{i=1}^{n} \rho(U_i) \, F(X) \, d\mu_{C_s(X)} \end{aligned} \tag{3.20}$$

The sum is over partitions $\{U_\gamma\}$ of X with $U_\gamma = I_\gamma \times Y_\gamma$, Y_γ connected, and orderings $(U_1,..., U_n)$ of these with U_1 always the one containing the first unit line in X. The term with $\{U_\gamma\} = \{X\}$, if it occurs, is interpreted as $\int \rho(X)\, F(X)\, d\mu_{C_s(X)}$. This is our cluster expansion.

3.5. Estimates

We want to show that $K_F(X)$ is exponentially small if X is large or diffuse. The main ingredients are the exponential decay bound (2.6) for C and bounds of the form of Lemma 3.1 for ρ.

For any paved set X, let $|X|$ be the number of elements $I \times \{x\}$ in X. With the intervals I represented by their midpoint, let $d(X, X')$ be the distance between two such sets X, X', and let $\mathscr{L}(X)$ be the length of the shortest tree joining the elements of X.

Theorem 3.2. Let F be a polynomial which factors over paved sets. For $a \geqslant 1$, let m^2 be sufficiently large and let λ be sufficiently small so $\lambda m^2 \leqslant 1$. Then for any X, $|K_F(X)| \leqslant \mathcal{O}(1)$, while for $|X| \geqslant 2$ we have

$$|K_F(X)| \leqslant [\mathcal{O}(m^{-1})]^{|X|-1} \exp[-a\mathscr{L}(X)] \tag{3.21}$$

The bounds are uniform in the volume $|\Lambda|$, and in the polynomial F if the degree and coefficients are bounded.

As preparation we have the following result.

Lemma 3.3. Under the same hypotheses, given an ordered partition $(U_1, ..., U_n)$ of X as in (3.20) $(n \geqslant 2)$ we have

$$\left| \int \prod_{j=2}^{n} [C(U_j, U_{\eta(j)}) \circ \Delta_\varphi] \prod_{i=1}^{n} \rho(U_i)\, F(X)\, d\mu_{C_s}(X) \right|$$
$$\leqslant [\mathcal{O}(m^{-1})]^{|X|-1} \prod_{j=2}^{n} \exp[-\mathrm{ad}(U_j, U_{\eta(j)})] \prod_{i=1}^{n} \exp(-a\,|U_i|) \tag{3.22}$$

Proof. We first take apart the $\rho(U_j) = \rho(I_j \times Y_j)$ by (3.4). The result is a sum over collections of bonds $\{b_\alpha\}$ such that the connected sets they determine are exactly $Y_1, ..., Y_n$. For each such $\{b_\alpha\}$ the summand is then

$$\int d\mu_{C_s(X)} \prod_j [C(U_j, U_{\eta(j)}) \circ \Delta_\varphi]$$
$$\times \prod_i \left[\left\{ \prod_{b_\alpha \subset Y_i} [\exp(-V_{I_i, b_\alpha}) - 1 \right\} \left[\prod_{x \in Y_i} \exp(-V_{I_i, x}) \right] F(U_i) \right] \tag{3.23}$$

It suffices to show that (3.23) satisfies the bound of the lemma because the sum over $\{b_\alpha\}$ contributes a factor $\prod_i \exp[\mathcal{O}(1)\,|Y_i|] = \exp[\mathcal{O}(1)\,|X|]$ and this does not affect the bound.

We next distribute the formal derivatives over the factors in the product. Let $\mathscr{K}=(2,..., n)\times(0, 1)$ and for $\underline{t}=(t_k)_{k\in\mathscr{K}}$ and $\underline{x}=(x_k)_{k\in\mathscr{K}}$ define

$$\mathscr{C}(\underline{t}, \underline{x})=\prod_j C(U_j, U_{\eta(j)})_{x_{j,0}, x_{j,1}}(t_{j,0}, t_{j,1})$$

so that

$$\prod_j [C(U_j, U_{\eta(j)})\circ \Delta_\varphi]=\int \underline{dt}\, \underline{dx}\, \mathscr{C}(\underline{t}, \underline{x}) \prod_{k\in\mathscr{K}} \partial/\partial\varphi_{x_k}(t_k)$$

where $\underline{dx}$ means counting measure. Then (3.23) may be written

$$\begin{aligned}\sum_\gamma \int d\mu_{C_s}(X) \int \underline{dt}\, \underline{dx}\, \mathscr{C}(\underline{t}, \underline{x})\\ \times\prod_i \Bigg[\Bigg\{\prod_{b_\alpha\subset Y_i} \partial^{\gamma^{-1}(i, b_\alpha)}[\exp(-V_{I_i, b_\alpha})-1]\Bigg\}\\ \times\Bigg[\prod_{x\in Y_i} \partial^{\gamma^{-1}(i,x)} \exp(-V_{I_i, x})\Bigg] [\partial^{\gamma^{-1}(i)} F(U_i)]\Bigg] \qquad (3.24)\end{aligned}$$

Here the sum is over all functions γ from $\mathscr{K}$ to $\mathscr{I}$, where $\mathscr{I}=\mathscr{I}_1\cup\cdots\cup\mathscr{I}_n$ and

$$\mathscr{I}_i=\{(i, b_\alpha): b_\alpha\subset Y_i\}\cup\{(i, x): x\in Y_i\}\cup\{i\}$$

Actually, not all γ's contribute to this sum. If we define $\mathscr{K}_1=\bigcup_{j\in\eta^{-1}(1)} (j, 1)$ and for $2\leqslant i\leqslant n$

$$\mathscr{K}_i=(i, 0)\cup\Bigg[\bigcup_{j\in\eta^{-1}(i)} (j, 1)\Bigg]$$

then the derivatives $\partial/\partial\varphi_{x_k}(t_k)$, $k\in\mathscr{K}_i$, are localized in U_i and so must act on functions localized in U_i. Thus, we must have $\gamma(\mathscr{K}_i)\subset\mathscr{I}_i$.

The derivatives further distributed themselves according to partitions P_{i,b_α} of $\gamma^{-1}(i, b_\alpha)$ and $P_{i,x}$ of $\gamma^{-1}(i, x)$. Let $P=(\{P_{i,b_\alpha}\}, \{P_{i,x}\}, \{\gamma^{-1}(i)\})$ be the induced partition of $\mathscr{K}$. Then (3.24) may be written

$$\sum_\gamma \sum_P \int d\mu_{C_s}(X) \int \underline{dt}\, \underline{dx}\, \mathscr{C}(\underline{t}, \underline{x})\, M(\underline{t}, \underline{x}) N \qquad (3.25)$$

Here we have defined

$$M(\underline{t}, \underline{x}) = \prod_i \Bigg[\Bigg(\prod_{b_\alpha \subset Y_i} \prod_{\pi \in P_{i,b_\alpha}} \partial^\pi(-V_{I_i,b_\alpha}) \Bigg) \times \Bigg(\prod_{x \in Y_i} \prod_{\pi \in P_{i,x}} \partial^\pi(-V_{I_i,x}) \Bigg) [\partial^{\gamma^{-1}(i)} F(U_i)] \Bigg]$$

For each i, the products over b_α and x are restricted to $\gamma^{-1}(i, b_\alpha) \neq \varnothing$ and $\gamma^{-1}(i, x) \neq \varnothing$. We have also defined

$$N = \prod_i \Bigg[\Bigg\{ \prod_{b_\alpha \subset Y_i} [\exp(-V_{I_i,b_\alpha}) - \delta_{\gamma^{-1}(i,b_\alpha),\varnothing}] \Bigg\} \Bigg(\prod_{x \in Y_i} \exp(-V_{I_i,x}) \Bigg) \Bigg]$$

The derivatives may now be evaluated. For $t_k \in I$ (the only case that occurs) we have

$$\frac{\partial V_{I,x}}{\partial \varphi_{x_k}(t_k)} = \Bigg[\Bigg(-\frac{3}{2} \lambda \varphi_{x_k}(t_k) + \lambda(m^2 + 2d)\, \varphi^3_{x_k}(t_k) + \frac{3}{4} \lambda^2 \varphi^5_{x_k}(t_k) \Bigg) + \Bigg(\frac{\lambda}{2} \varphi^3_{x_k}(t_k) + \lambda q'(\varphi_{x_k}(t_k)) \Bigg) \delta(t_k) + \Bigg(\frac{\lambda}{2} \varphi^3_{x_k}(t_k) \Bigg) \delta(t_k - T) \Bigg] \delta_{x,x_k}$$

$$\frac{\partial^2 V_{I,x}}{\partial \varphi_{x_k}(t_k)\, \partial \varphi_{x_l}(t_l)} = \Bigg[\Bigg(-\frac{3}{2} \lambda + 3\lambda(m^2 + 2d)\, \varphi^2_{x_k}(t_k) + \frac{15}{4} \lambda^2 \varphi^5_{x_k}(t_k) \Bigg) + \Bigg(\frac{3}{2} \lambda \varphi^2_{x_k}(t_k) + \lambda q''(\varphi_{x_k}(t_k)) \Bigg) \delta(t_k) + \Bigg(\frac{3}{2} \lambda \varphi^2_{x_k}(t_k) \Bigg) \delta(t_k - T) \Bigg] \delta_{x,x_k} \delta_{x_k,x_l} \delta(t_k - t_l)$$

and so forth; and for $b = (x, y)$ we have

$$\frac{\partial V_{I,b}}{\partial \varphi_{x_k}(t_k)} = \frac{3}{4} \lambda \varphi^2_x(t_k)\, \varphi_y(t_k) \delta_{x_k,x} + \frac{\lambda}{4} \varphi^3_x(t_k) \delta_{x_k,y}$$

and so forth.

With these evaluations we can break up M into a sum over monomials. We have

$$M(\underline{t}, \underline{x}) = \sum_{\omega} c_\omega M_\omega(\underline{t}, \underline{x})\, \delta_\omega(\underline{t}, \underline{x})$$

Here ω runs over some index set which we will not need to specify explicitly. The c_ω are constants [they collect factors $-\frac{3}{2}\lambda$, $\lambda(m^2+2d)$, etc.], the M_ω are monomials in $\varphi_{x_k}(t_k)$, and δ_ω collects all the δ-functions. In the time variables the δ-functions identify the t_k with themselves or 0 or T, and in the space variables the δ-functions identify the x_k with themselves or some other $x \in X$.

Now (3.25) is written

$$\sum_{\gamma} \sum_{P} \sum_{\omega} \int d\mu_{C_s(X)} \int d\lambda_\omega(\underline{t}, \underline{x}))\mathscr{C}(\underline{t}, \underline{x})\, c_\omega M_\omega(\underline{t}, \underline{x}) N \tag{3.26}$$

where $d\lambda_\omega(\underline{t}, \underline{x}) = \delta_\omega(\underline{t}, \underline{x})\, \underline{dt}\, \underline{dx}$ is a measure on $\mathbb{R}^{2n} \times \Lambda^{2n}$.

Now we begin the estimates. A key role is played by the numbers $d_i = |\mathscr{K}_i|$ which are the incidence numbers for the tree graph η.

First, in the integral over φ, apply the Schwarz inequality to separate M_ω and N. We have then

$$\int M_\omega^2\, d\mu_{C_s(X)} \leqslant \prod_i N(U_i)!\, \exp[\mathcal{O}(1)\, N(U_i)] \tag{3.27}$$

where $N(U_i)$ is the number of variables $\varphi_{x_k}(t_k)$ in M_ω^2 with $(t_k, x_k) \in U_i$. Bounds of this type are standard; see ref. 13, Theorem 8.5.5. The main ingredient is the exponential decay of $C_s(X)$, which follows from the exponential decay of C. (Usually the role of the U_i is played by unit blocks, but the more general case is easily deduced from the latter.)

The number of variables in U_i for M_ω is at most deg $F(U_i)$ plus the number of derivatives acting in U_i, namely d_i, times the maximum of φ's introduced by a differentiation, namely $r \equiv \max(5, \deg q - 1)$. Thus, we have $N(U_i) \leqslant 2[\deg F(U_i) + r d_i]$. If we also use $\sum_i d_i = 2n-2$, we obtain

$$\left(\int M_\omega^2\, d\mu_{C_s(x)}\right)^{1/2} \leqslant \prod_i \mathcal{O}(1)(d_i!)^r \tag{3.28}$$

We also have, by (3.7) and (3.8),

$$\left(\int N^2\, d\mu_{C_s(X)}\right)^{1/2} \leqslant \prod_{\substack{i, b_\alpha \\ \gamma^{-1}(i, b_\alpha) = \varnothing}} [\mathcal{O}(m^{-2})] \prod_i \exp[\mathcal{O}(1)\, |Y_i|] \tag{3.29}$$

For c_ω note that the coefficients in $V_{I,x}$ are at worst $\mathcal{O}(1)$, while those in $V_{I,b}$ are $\mathcal{O}(\lambda) \leqslant \mathcal{O}(m^{-2})$. This lead to the bound

$$|c_\omega| \leqslant \prod_{\substack{i, b_\alpha \\ \gamma^{-1}(i, b_\alpha) \neq \varnothing}} [\mathcal{O}(m^{-2})] \prod_i \mathcal{O}(1) \tag{3.30}$$

Combining (3.28)–(3.30) and using $\prod_{i, b_\alpha} \mathcal{O}(m^{-2}) \leqslant \prod_i [\mathcal{O}(m^{-2})]^{|Y_i| - 1}$, we obtain

$$\int |c_\omega M_\omega N| \, d\mu_{C_s(X)} \leqslant \prod_i [\mathcal{O}(m^{-2})]^{|Y_i| - 1} \exp[\mathcal{O}(1) \, |Y_i|](d_i!)^r \tag{3.31}$$

We next claim that

$$\int_{\text{supp}\, \mathscr{C}} d\lambda_\omega(\underline{t}, \underline{x}) \leqslant 1$$

The time integral can be written in the form $\int \prod_k (\Delta_k \, dt_k)$, where Δ_k is one of 1, $\delta(t_k)$, $\delta(t_k - T)$, or $\delta(t_k - t_{k'})$ for some $k' < k$ in a lexicographic ordering of $\mathscr{K}$. Since $\int \Delta_k \, dt_k \leqslant 1$ (supp $\mathscr{C}$ has unit intervals), if we do the integrals in reverse order, we have $\int \prod_k (\Delta_k \, dt_k) \leqslant 1$. Similarly, the sum over space variables is bounded by 1.

Using this result and (2.6), we obtain

$$\left| \int \mathscr{C}(\underline{t}, \underline{x}) \, d\lambda_\omega(\underline{t}, \underline{x}) \right| \leqslant \|\mathscr{C}\|_\infty \leqslant \prod_j \mathcal{O}(m^{-2}) \exp[-2ad(U_j, U_{\eta(j)})] \tag{3.32}$$

Now we estimate the number of terms in the sums over γ, P, ω. Since γ takes each $\mathscr{K}_i$ with d_i elements to $\mathscr{I}_i$ with $\mathcal{O}(1) \, |Y_i|$ elements, the number of γ's is less than $\prod_i [\mathcal{O}(1) \, |Y_i|]^{d_i} \leqslant \prod_i (d_i)! \exp[\mathcal{O}(1) \, |Y_i|]$. For each γ, we estimate the number of partitions P_{i, b_α} of $\gamma^{-1}(i, b_\alpha)$ by $|\gamma^{-1}(i, b_\alpha)|!$, etc. Then the number of partitions P is bounded by

$$\begin{aligned} &\prod_i \left[\prod_{b_\alpha} |\gamma^{-1}(i, b_\alpha)|! \prod_x |\gamma^{-1}(i, x)|! \right] \\ &\quad \leqslant \prod_i \left[\sum_{b_\alpha} |\gamma^{-1}(i, b_\alpha)| + \sum_x |\gamma^{-1}(i, x)| \right]! \\ &\quad \leqslant \prod_i d_i! \end{aligned}$$

Finally, for each γ, P the number of V's and F's in M is less than $2n$ and each contributes $\mathcal{O}(1)$ terms to the sum over ω. Thus, the sum over ω has less than $[\mathcal{O}(1)]^{2n} \leqslant \prod_i \exp[\mathcal{O}(1) \, |Y_i|]$ terms.

Combining all the above and using $|Y_i| = |U_i|$, we have that (3.26) is bounded by

$$\prod_{j=2}^{n} \{\mathcal{O}(m^{-2}) \exp[-2ad(U_j, U_{\eta(j)})]\}$$
$$\times \prod_{i=1}^{n} \{\mathcal{O}(m^{-2})^{|U_i|-1} \exp[\mathcal{O}(1)\,|U_i|](d_i!)^{r+2}\} \tag{3.33}$$

But if d_i is large, some of the $d(U_j, U_{\eta(j)})$ for $\eta(j) = i$ must be large and one can show

$$\prod_i (d_i!)^{\mathcal{O}(1)} \leqslant \prod_j \exp[\mathcal{O}(1)\, d(U_j, U_{\eta(j)})] \tag{3.34}$$

(see ref. 13, proof of Lemma 18.7.2 for the idea). One may assume that a is large enough so that this factor is controlled by $\prod_j \exp[-ad(U_j, U_{\eta(j)})]$. One also has $(\sum_i |U_i| - 1$ factors of $\mathcal{O}(m^{-2})$. This is equal to $|X| - 1$, so we may use half of them to get a factor $[\mathcal{O}(m^{-1})]^{|X|-1}$. It is also greater than $\frac{1}{2}(\sum_i |U_i|)$, so we can identify a factor $\prod_i [\mathcal{O}(m^{-1/2})]^{|U_i|}$, which is less than $\prod_i \exp(-a|U_i|)$ for m sufficiently large. Thus, (3.33) is dominated by the right side of (3.22), as required.

Proof of Theorem 3.2. The proof is now more or less standard. If $|X| \geqslant 2$ and $X \neq I \times Y$, Y connected, then by Lemma 3.3 (with $3a$ instead of a) we have

$$|K_F(X)| \leqslant \sum_{\{U_\gamma\}} \sum_{(U_1,\ldots,U_n)} \sum_\eta \int ds\, f(\eta, s)$$
$$\times [\mathcal{O}(m^{-1})]^{|X|-1} \prod_{j=2}^{n} \exp[-3ad(U_j, U_{\eta(j)})] \prod_{i=1}^{n} \exp(-3a\,|U_i|) \tag{3.35}$$

The result now follows by the following steps.

1. Use $\sum_j d(U_j, U_{\eta(j)}) - \sum_i |U_i| \geqslant \mathscr{L}(X)$ to get a factor $\exp[-a\mathscr{L}(X)]$.

2. Take a factor $\prod_i \exp(-a\,|U_i|) = \prod_\gamma \exp(-a\,|U_\gamma|)$ outside the sum over $(U_1,\ldots, U_n)$. The rest is estimated by

$$\sum_{(U_1,\ldots,U_n)} \sum_j \exp[-ad(U_j, U_{\eta(j)})] \prod_i \exp(-a\,|U_i|)$$
$$\leqslant \{\exp[\mathcal{O}(1) n]\} \prod_i (d_i - 1)! \tag{3.36}$$

To see that this is so, estimate the sum over U_n by

$$\sum_{U_n} \exp[-ad(U_n, U_{\eta(n)}) - a\,|U_n|] \leqslant \mathcal{O}(1)\,|U_{\eta(n)}|$$

At the jth level we have

$$\begin{aligned}\sum_{U_j} \exp[-ad(U_j, U_{\eta(j)}) - a\,|U_j|]\,|U_j|^{d_j-1} &\leqslant \sum_{U_j} \exp[-ad(U_j, U_{\eta(j)})](d_j-1)! \\ &\leqslant \mathcal{O}(1)\,|U_{\eta(j)}|\,(d_j-1)!\end{aligned}$$

At the last step there is no sum over U_1 and we just have $|U_1|^{d_1} \exp(-a\,|U_1|) \leqslant \mathcal{O}(1)(d_1-1)!$.

3. For the sum over η use the bound[17]

$$\sum_{\eta} \int ds\, f(\eta, s) \prod_i (d_i-1)! \leqslant e^{\mathcal{O}(1)n}$$

[Alternatively, one can use (3.34) to gain a factor $\prod_i (d_i!)^{-1}$ in the lemma and then use the more elementary bound $\sum_\eta \int ds\, f(\eta, s) \leqslant e^{\mathcal{O}(1)n}$.]

4. The sum over partitions $\{U_\gamma\}$ is controlled by

$$\sum_{\{U_\gamma\}} \prod_\gamma e^{-a|U_\gamma|} \leqslant e^{\mathcal{O}(1)|X|}$$

The $e^{\mathcal{O}(1)n}$ factors also have this bound and all are absorbed by $[\mathcal{O}(m^{-1})]^{|X|-1}$. Thus, the proof is complete in this case.

If $X = I \times Y$, there is an extra term with $\{U_\gamma\} = X$ which has a special definition. If also $|X| \geqslant 2$, use Lemma 3.1 to bound this by $[\mathcal{O}(m^{-2})]^{|X|-1}$. Since $|X| - 1 = \mathscr{L}(X)$ for connected X, the result follows in this case. If $|X| = 1$, so $X = I \times \{x\}$, the extra term is the only term and the bound $|K_F(X)| \leqslant \mathcal{O}(1)$ follows from $|\rho_{I,x}| \leqslant \mathcal{O}(1)$. This completes the proof.

4. CONSEQUENCES

The cluster expansion leads directly to estimates on the decay of correlations. These are expressed in terms of connected expectations (truncated expectations, cummulants) defined by

$$E^c(f_i, ..., f_n) = \partial^n/\partial\alpha_1 \cdots \partial\alpha_n \log\left[\int \prod_i (1 + \alpha_i f_i) e^{-V_T}\, d\mu_C\right]\Bigg|_{\alpha=0} \qquad (4.1)$$

These are combinations of the ordinary expectations (2.22), for example,

$$E^c(f_1, f_2) = E(f_1 f_2) - E(f_1)\, E(f_2) \tag{4.2}$$

The main result is the following:

Theorem 4.1. Given $a \geqslant 1$, let m^2 be sufficiently large and λm^2 sufficiently small. Then for $(t_k, x_k) \in \Delta_k \equiv I_k \times \{x_k\}$ we have uniformly in the volume the tree decay

$$|E^c(\varphi_{x_1}(t_1),..., \varphi_{x_n}(t_n))| \leqslant \mathcal{O}(m^{-1}) \exp[-a\mathscr{L}(\Delta_1 \cup \cdots \cup \Delta_n)]$$

Proof. Take T sufficiently large and assume $t_i \in [0, T]'$, so we may work on Ω' as in the previous section; the case of integral t_i follows by limits. The proof that follows is mostly standard.[14,18] Let $F = \prod_k [1 + \alpha_k \varphi_{x_k}(t_k)]$. By the cluster expansion (3.19),

$$\int F e^{-V_T}\, d\mu_C = \sum_{\{X_i\}} \prod_i K_F(X_i) \tag{4.3}$$

Let $\Delta = I \times \{x\}$ denote a unit line. Factoring out the unit lines, we can write the right side of (4.3).

$$\prod_{\Delta \subset \Lambda} K_F(\Delta) \sum_{\{X_i\}} \prod_i \tilde{K}_F(X_i) \tag{4.4}$$

where now the sum is over collections of disjoint sets $\{X_i\}$ with $|X_i| > 2$, and where

$$\tilde{K}_F(X) = K_F(X) \prod_{\Delta \subset X} K_F(\Delta)^{-1} \tag{4.5}$$

Next take the logarithm, treating the expansion as the partition function for a gas of (disconnected) polymers. This takes the form

$$\log\left(\int F e^{-V_T}\, d\mu_C\right)$$
$$= \sum_{\Delta \subset \Lambda} \log K_F(\Delta) + \sum_{n=1}^{\infty} 1/n! \sum_{(X_1,..., X_n)} a(X_1,..., X_n) \prod_{i=1}^{n} \tilde{K}_F(X_i) \tag{4.6}$$

where the sum is over ordered n-tuples $(X_1,..., X_n)$ of paved sets and $a(X_1,..., X_n) = 0$ unless they overlap.

It is straightforward to show that under our hypotheses and for $|\alpha| = \sup_i |\alpha_i|$ sufficiently small

$$|K_F(\Delta) - 1| \leqslant \mathcal{O}(\lambda m^2) + \mathcal{O}(|\alpha|) < \tfrac{1}{2} \tag{4.7}$$

and so from Theorem 3.2, for $|X| \geqslant 2$,

$$|\tilde{K}_F(X)| \leqslant [\mathcal{O}(m^{-1})]^{|X|-1} \exp[-2a\mathscr{L}(X)] \tag{4.8}$$

It follows that

$$\sup_{\Delta} \sum_{X \supset \Delta} |\tilde{K}_F(X)| \, e^{|X|} \leqslant \mathcal{O}(m^{-1}) \tag{4.9}$$

This estimate is sufficient to ensure that the sum in (4.6) converges and is bounded by $\mathcal{O}(m^{-1}) \, |\Lambda|$.

Now suppose that not all (t_k, x_k) are on the same line Δ. (This case can be included by a separate argument.) Then $\partial^n/\partial\alpha_1,..., \partial\alpha_n[\cdots]_{\alpha=0}$ gives zero on the first sum in (4.6). For the second sum only those terms $(X_1,..., X_n)$ with the cover property $\bigcup_k \Delta_k \subset \bigcup_i X_i$ survive [otherwise some (t_k, x_k) is outside $\bigcup_i X_i$, the term is independent of α_k, and the derivative gives zero]. Thus, we have

$$E^c(\varphi_{x_1}(t_1),..., \varphi_{x_n}(t_n))$$
$$= \partial^n/\partial\alpha_1,..., \partial\alpha_n \left[\sum_{n=1}^{\infty} 1/n! \sum_{(X_1,..., X_n)}' a(X_1,..., X_n) \prod_i \tilde{K}_F(X_i) \right]\Bigg|_{\alpha=0} \tag{4.10}$$

where the prime indicates that the cover property must be satisfied.

To estimate this, take a factor $\prod_i \exp[-a\mathscr{L}(X_i)]$ from the estimate on $\prod_i \tilde{K}_F(X_i)$. By the connectivity of the X_i this is less than $\exp[-a\mathscr{L}(X_1 \cup \cdots \cup X_n)]$, which by the cover property is less than $\exp[-a\mathscr{L}(\Delta_1 \cup \cdots \cup \Delta_n)]$. The rest of the estimate proceeds as before, except that the cover property also eliminates a sum over the whole volume. The bracketed expression in (4.10) is then bounded uniformly in the volume by

$$[\cdots] \leqslant \mathcal{O}(m^{-1}) \exp[-a\mathscr{L}(\Delta_1 \cup \cdots \cup \Delta_n)] \tag{4.11}$$

The bound holds in a complex polydisk $|\alpha_i| < R$ with $R^{-1} \leqslant \mathcal{O}(1)$, and so by the Cauchy bounds we have our result.

Remark. Theorem 3.2 required $\lambda m^2 \leqslant 1$ and Theorem 4.1 needed that λm^2 was sufficiently small. In fact, these conditions on λ are unnecessary: λ can be anything as long as m^2 is sufficiently large.

To see this, make the change of variables $\varphi \to m^{-\alpha}\varphi$ in the correlation function (2.22), say with $Q = 0$ for simplicity. Then we replace P, W, C by new objects

$$\tilde{P}=\sum_x \frac{1}{4}\lambda m^{-4\alpha}\varphi_x^4$$

$$\tilde{W}=\sum_x\left(-\frac{3}{4}\lambda m^{-2\alpha}\varphi_x^2+\frac{1}{4}\lambda m^{-4\alpha}\varphi_x^3(D\varphi)_x+\frac{1}{8}\lambda^2 m^{-6\alpha}\varphi_x^6\right)$$

$$\tilde{C}=m^{2\alpha}C$$

Now, if $1/2<\alpha<1$, all the coefficients in $\tilde{P}$ and $\tilde{W}$ are small for m large (the local quartic term is now $\frac{1}{4}\lambda m^{2-4\alpha}\varphi_x^4$) and $\tilde{C}$ has exponential decay as in (2.6) with a small coefficient in front [now $\mathcal{O}(m^{2\alpha-2})$ instead of $\mathcal{O}(m^{-2})$]. These features are all we need for the proofs to go through.

As an application, we show that the correlation functions approach equilibrium as time evolves.

Corollary 4.2. Uniformly in the volume, as $t\to\infty$,

$$E(\varphi_{x_1}(t_1+t)\cdots\varphi_{x_n}(t_n+t))$$
$$=E^*(\varphi_{x_1}(t_1)\cdots\varphi_{x_n}(t_n))+\mathcal{O}(e^{-at}) \tag{4.12}$$

Proof. Let $f_k=\varphi_{x_k}(t_k+t)$. Since E^* is stationary, the difference between the expectations can be written $E(f_1\cdots f_n)-E^*(f_1\cdots f_n)$. Also, it suffices to prove the result for the connected functions, since the ordinary correlation function are sums of products of these.

Let E_α denote the expectation with Q replaced by αQ in (2.21) and (2.22). Then $E_0=E^*$ and $E_1=E$ and we have

$$E^c(f_1,\dots,f_n)-E^{*,c}(f_1,\dots,f_n)$$
$$=\int_0^1 d/d\alpha\, E_\alpha^c(f_1,\dots,f_n)\,d\alpha$$
$$=\int_0^1 E_\alpha^c(f_1,\dots,f_n,\,-Q)\,d\alpha$$
$$=\sum_x\int_0^1 E_\alpha^c(f_1,\dots,f_n,\,-\lambda q(\varphi_x(0))) \tag{4.13}$$

By the theorem (actually a slight modification allowing powers of the fields)

$$|E_\alpha^c(f_1,\dots,f_n,\,-\lambda q(\varphi_x(0)))|\leqslant\mathcal{O}(m^{-1})\exp[-a\mathcal{L}(x_1,\dots,x_n,x)]\exp(-at)$$

The first factor gives the uniform convergence of the sum over x and so $|(4.13)|\leqslant\mathcal{O}(e^{-at})$.

Remarks. (1) Since the results are uniform in the volume, they also hold for any infinite-volume limit. Of course, cluster expansions can also be used to obtain infinite-volume limits.

(2) With no essential changes, one could replace φ^4 in the original action (1.1) by any lower-semibounded polynomial.

(3) We have treated models in which the correlation length is small (here bounded by a^{-1}). It would also be of considerable interest to study critical theories with an infinite correlation length. Formal treatments of such questions by renormalization group methods are given in refs. 1 and 19. It may be possible to make these rigorous using the methods introduced by Gawedski and Kupiainen.[20]

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